

The \$-game

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Abstract. We propose a payoff function extending Minority Games (MG) that captures the competition between agents to make money. In contrast with previous MG, the best strategies are not always targeting the minority but are shifting opportunistically between the minority and the majority. The emergent properties of the price dynamics and of the wealth of agents are strikingly different from those found in MG. As the memory of agents is increased, we find a phase transition between a self-sustained speculative phase in which a “stubborn majority” of agents effectively collaborate to arbitrage a market-maker for their mutual benefit and a phase where the market-maker always arbitrages the agents. A subset of agents exhibit a sustained non-equilibrium risk-return profile.

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The Minority Game (MG)[1] is perhaps the simplest in the class of multi-agent games of interacting inductive agents with limited abilities competing for scarce resources. Many published works on MG have motivated their study by their relevance to financial markets, because investors exhibit a large heterogeneity of investment strategies, investment horizons, risk aversions and wealths, have limited resources and time to dedicate to novel strategies and the minority mechanism is found in markets. Here, our goal is to point out that the minority mechanism is a relatively minor contribution to the self-organization of financial markets. We develop a better description based on a financially motivated payoff function. Following the standard specification of MG, we assume that markets are purely speculative, that is, agents profit only from changes in the stock price. In addition, agents are chartists or technical analysts who only analyze past realization of prices, with no anchor on fundamental economic analysis.

A MG is a repeated game where N players have to choose one out of two alternatives at each time step based on information represented as a binary time series $B(t)$. Those who happen to be in the minority win. Each agent i possesses a memory of the last m digits of $B(t)$. A strategy gives a prediction for the next outcome of $B(t)$ based on the history of the last m digits of B . Since there are 2^m possible histories, the total number of strategies is given

by $S = 2^{2^m}$. Each agent holds the same number s of (but in general different) strategies among the S possible strategies. At each time t , every agent uses her most successful strategy (in terms of payoff, see below) to decide whether to buy or sell an asset. The agent takes an action $a_i(t) = \pm 1$ where 1 is interpreted as buying an asset and -1 as selling an asset. The excess demand, $A(t)$, at time t is therefore given as $A(t) = \sum_{i=1}^N a_i(t)$. The payoff of agent i in the MG is given by:

$$g_i(t) = -a_i(t)A(t). \quad (1)$$

As the name of the game indicates, if a strategy i is in the minority ($a_i(t)A(t) < 0$), it is rewarded. In other words, agents in MG try to be anti-imitative. To ensure causality, the notation $-a_i(t)A(t)$ in (1) must be understood as $-a_i(t - 1/2)A(t)$ since the actions/strategies of the agents take place *before* the price (and thus the payoff) can be determined. The richness and complexity of minority games stem from the fact that agents have to be different; theories based on an effective representative agent are bound to fail because she would represent the majority. MG are intrinsically frustrated and fluctuations and heterogeneities are the key ingredients.

In order to model financial markets, several authors have used the following or slight variants of the following equation for the return $r(t)$ [2,3]

$$r(t) \equiv \ln(p(t)) - \ln(p(t-1)) = A(t)/\lambda, \quad (2)$$

where $\lambda \propto N$ is the liquidity. The fact that the price goes in the direction of the sign of the order imbalance $A(t)$

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is well-documented [4–9]. By constructing and analyzing a large database of estimated market-wide order imbalances for a comprehensive sample of NYSE stocks during the period 1988–1998 inclusive, Chordia *et al.* [10] confirm that contemporaneous order imbalance $A(t)$ exerts an extremely significant impact on market returns in the expected direction; the positive coefficients of their regressions imply that excess buy (sell) orders drive up (down) prices, in qualitative agreement with (2).

Let us assume that an agent thinks at time $t - 1/2$ that the unknown future price $p(t)$ will be larger than the known previous quote $p(t - 1)$ and larger than the next future quote $p(t + 1)$, thus identifying $p(t)$ as a local maximum. Her best strategy is to put a sell order at time $t - 1/2$ in order for the sale to be realized at time t at the local price maximum, allowing her to profit from future drops at later times. She will then profit and cash in the money equal to the drop from the local maximum at time t to a smaller price realized at $t + 1$ or later. In this case, the optimal strategy is thus to be in the minority as seen from the relation between the direction of the price change given by the sign of $r(t)$ and the direction of the majority given by the sign of $A(t)$. Alternatively, if the agent thinks at time $t - 1/2$ that $p(t - 1) < p(t) < p(t + 1)$, her best strategy is to put a buy order at time $t - 1/2$, realized at the price $p(t)$ at time t . She will then profit by the amount $p(t + 1) - p(t)$ if her expectation that $p(t) < p(t + 1)$ is born out. In this case, it is profitable for an agent to be in the majority, because the price continues to go up, driven by the majority, as seen from (2). In order to know when the price reaches its next local extremum and optimize their gains, the agents need to predict the price movement over the next *two* time steps ahead (t and $t + 1$), and not only over the next time step as in the standard MG. This pinpoints the fundamental misconception of MG as models of financial markets. Indeed, by shifting from minority to majority strategies and vice-versa, an agent tries at each time step to gain $|p(t + 1) - p(t)|$ whatever the sign of $p(t + 1) - p(t)$: an ideal strategy is a “return rectifier.” Because an agent’s decision $a(t - 1/2)$ at time $t - 1/2$ is put into practice and invested in the stock market at time t , the decision will bring its fruit from the price variation from t to $t + 1$. From (2), this price variation is simply proportional to $A(t)$. Therefore, the agent has a positive payoff if $a(t - 1/2)$ and $A(t + 1/2)$ have the same sign. As a consequence, in the spirit of the MG (and using the MG notation without half-time scales), the correct payoff function is¹

$$g_i^{\$}(t + 1) = a_i(t)A(t + 1). \quad (3)$$

The superscript \$ is a reminder that the action taken by agent i at time t results at time $t + 1$ in a percentage gain/loss of $g_i^{\$}(t + 1)/\lambda$ (see (2)). We will refer to the game where the agents use (3) as the “\$-game” since, by

¹ A similar rule for the update of scores was recently considered in another model [11] but with a different sign. After appearance of our present paper in *cond-mat*, we were notified by the authors of [11] that their sign difference was a misprint, so that ours and their rule are the same.

using this payoff function, the agents strive to increase their wealth. This reasoning stresses that, in real markets, the driving force underlying the competition between investors is not a struggle to be in the minority at each time step, but rather a fierce competition to gain money.

In reference [12], Marsili presents an interesting derivation of the minority game based on a reasonable approximation of market mechanisms by emphasizing the role of agents’ expectations. By playing with the nature of the agents’ expectation, Marsili also shows that the majority rule can emerge naturally and he studies mixed minority-majority games to find that, in both a minority and a majority game, expectations are self-fulfilled. The difference with our present work is multifold. First, Marsili postulates beliefs that are of a very simple nature and imposes the fraction of trend-followers (majority players) and contrarians (minority players). This leads to different market regimes depending on this fraction. In contrast, our agents do not belong to fixed populations of either majority or minority players but any agent freely shifts from trend-follower to contrarian by using an adaptive behavior. Thus, Marsili’s paper emphasizes expectations at the cost of freezing the division between the two categories of trend followers and contrarians. We do not use expectations but only the objective of maximizing a payoff in order to address the problem of adaptation leading to possible shifts between the two classes of strategies. We believe that our approach is more relevant to understanding concretely real markets. There are many evidences well-documented in the finance literature that investors may be mainly contrarian in certain phases of the market and become trend-followers in other phases (see for instance Ref. [13] in which Frankel and Froot found that, over the period 1981–1985, the market shifted away from the fundamentalists and towards the chartists to fuel the speculative bubble on the US dollar). Thus, rather than being either minority or majority players, our agents change adaptively from trend-followers to contrarians and *vice versa*. Our agents are thus both opportunistic majority and minority players, as they should to represent real investors.

In the simplest version of the model, each trade made by an agent is the exchange of one quanta of a riskless asset (cash) for one quanta of a risky one (asset) irrespective of the agent’s wealth or the price of the asset. The wealth of the i th agent at time t is given as

$$W_i(t) = N_i(t)p(t) + C_i(t), \quad (4)$$

where $N_i(t)$ is the number of assets held by agent i and $C_i(t)$ the cash possessed by agent i at time t . In order to illustrate the differences between the payoff functions (1) and (3), we have plotted in Figure 1 an example of the payoff (upper plot) of the best as well as the worst performing MG agent using (1). Each agent is allowed to take either a long or a short position, and we furthermore assume that the agents stay in the market at all times. This means that if *e.g.* an agent has taken a long position (*i.e.* taken the action $a_i = 1$ to buy a asset) the agent will not open new positions (and therefore does not contribute to the excess demand and price change) but keep the long

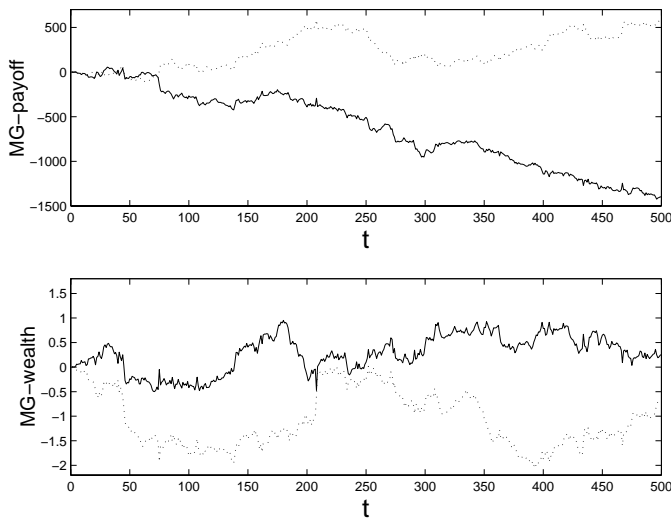


Fig. 1. Payoff function (1) (upper graph) and wealth (lower graph) for the MG-game showing the best (dotted line) and worst (solid line) performing agent for a game using $N = 501$ agents, memory of $m = 10$ and $s = 10$ strategies per agent. No transaction costs are applied.

position until she gets a signal to sell ($a_i = -1$) [14]. The lower plot of Figure 1 shows the wealth (4) corresponding to the agents of the upper plot. The consistently bad performance of the optimal MG-agent in terms of her wealth and reciprocally the relatively good performance for the worst MG-agent in terms of her wealth is a clear illustration of the fact that a minority strategy will perform poorly in a real market. This does not exclude however the potential usefulness of MG strategies in certain situations, in particular for identifying extrema, as discussed above and as illustrated recently in the prediction of extreme events [18]. In contrast, for the “\$-game” (3) presented here, the performance of the payoff function (3) matches by definition the performance of the wealth of the agents. The superficial observance by some MG of the stylized facts of financial time series is not a proof of their relevance and, in our opinion, express only the often observed fact that many models, be they relevant or irrelevant, can reproduce superficially a set of characteristics (see for instance a related discussion on mechanisms of power laws and self-organized criticality in chapters 14 and 15 of [19]).

In order for trading to occur and to fully specify the price trajectory, a clearing mechanism has to be specified. Here, we use a market maker who furnishes assets in case of demand and buys assets in case of supply [15]. The price fixing equation (2) implicitly assumes the presence of a market-maker, since the excess demand of the agents $A(t)$ always finds a counterpart. For instance, if the cumulative action of the agents is to sell 10 stocks, $A(t) = -10$, the market-maker is automatically willing to buy 10 stocks at the price given by (2). As pointed out in reference [15], expression (2) leads to an unbound market-maker inventory $S_M(t)$. In order to lower his inventory costs and the risk of being arbitrated, a market-maker will try to keep his inventory secret and in average close

to zero [10]. As shown in [15], this can be achieved by the following generalization of (2):

$$r(t) \equiv \ln(p(t)) - \ln(p(t-1)) = (A(t) - S_M(t))/\lambda, \quad (5)$$

with $S_M(t) = -\sum_{t=0}^{t-1} A(t)$. Expression (5) implies that, the larger is the long position the market-maker is holding, the more he will lower the price in order to attract buyers, and *vice versa* for a short position. Another way to ensure the same behavior is to introduce a spread or change the available liquidity [17].

We first study the price formation using (2) and resulting from a market competition between agents with payoff function (3) and compare it with the MG case (1) in the case with no constraint on the number of stocks held by each agent (*i.e.*, an agent can open a new position at each time step). Contrary to the MG case, we find that the price always diverges to infinity or goes zero within a few tens or hundreds of time steps. This behavior is observed for all values of N, m, s . Similar results are found if we replaced the price equation (2) with (5) which includes the market-maker strategy. The reason for this non-stationary behavior is that agents, using (3) as pay-off function, are able to collaborate to their mutual benefit. This happens whenever a majority among the agents can agree to “lock on” for an extended period of time to a common decision of either to keep on selling or buying. A constant sign of $A(t)$ is seen from either (2)-(4) or (3)-(5) to lead to a steady increase of the wealth of those agents sticking to the majority decision. A “stubborn majority” manages to collaborate by sticking to the same common decision – they all gain by doing so at the cost of the market-maker who is arbitrated. The mechanism underlying this cooperative behavior is the positive feedback resulting from a positive majority $A(t)$ which leads to an increase in the price (5) which in turn confirms the “stubborn majority” to stick to their decision and keep on buying, leading to a further confirmation of a positive $A(t)$. This situation is reminiscent of wild speculative phases in markets, such as occurred prior to the October 1929 crash in the US, before the 1994 emergent market crises in Asia, and more recently during the “new economy” boom on the Nasdaq stock exchange, in which margin requirements are decreased and/or investors are allowed to borrow more and more on their unrealized market gains. This situation is quite parallel to our model behavior in which agents can buy without restraint, pushing the prices up. Of course, some limiting process will eventually appear, often leading to the catastrophic stop of such euphoric phase.

We turn to the more realistic case where agents have bounded wealth, and study the limiting case where agents are allowed to keep only one long/short position at each time step. With this constraint, the previous positive feedback is no longer at work. Holding a position, an agent will contribute to future price changes only when she changes her mind. Thus, a “stubborn majority” can not longer directly influence future price changes through the majority term $A(t)$, but only now indirectly through the impact on the market maker strategy $S_M(t)$ in (5). Figure 2a show typical examples of price trajectories using (3)-(5) with

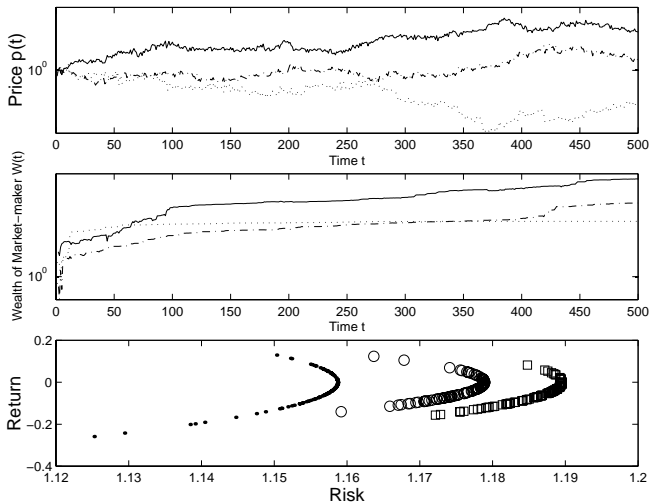


Fig. 2. Price, wealth of market-maker and risk-return plots for three different parameter choices using the payoff function (3) and the constraint that agents can only accumulate one position at a time. Solid line and black circle: $m = 10, s = 4$; dashed-dotted line and circle: $m = 10, s = 10$; dotted line and square: $m = 8, s = 10$.

agents keeping a single position (short/long) at any times, for three different choices of parameter values (N, m, s). The time series are quite similar to typical financial price series and possess their basic stylized properties (short-range correlation of returns, distribution of returns with fat tails, long-range correlation of volatility [20]). The corresponding wealth of the market maker is shown in Figure 2b. It exhibits a systematic growth, interrupted rarely for some short periods of time with small losses. The stochastic nature of the price trajectories is translated into an almost deterministic wealth growth for the market-maker, who is an almost certain winner (as it should and is in real market situations to ensure his survival and profitability). The market maker is similar to a casino providing services or entertainments and which profits from a systematic bias here resulting from the lack of cooperativity of the agents.

For each agent i , we define a risk parameter

$$R_i(t) = \langle (dW_i(t) - \langle dW_i \rangle_t)^2 \rangle_t \quad (6)$$

where $dW_i(t)$ is the change of wealth of agent i between t and $t - 1$. $R_i(t)$ is the volatility of the wealth of agent i . The average return per time step $\langle dW_i \rangle$ for each of the $N = 101$ different agents as a function of his volatility R_i is shown in Figure 2c (each point corresponds to one agent). Since agents choose either a short or a long position at each time step, a perfect performing agent is a return rectifier taking no risk. Similarly, the worst performing agent is consistently moving against the market, again with the risk defined from (6) equal to zero. This explains why the risk-return behavior seen in Figure 2c is an mirror image of the risk-return efficient frontier in Markovitz standard portfolio theory [16]. The figure shows that even though the market-maker arbitrages the agents as a group, some “clever” agents are still able to profit from their trade with

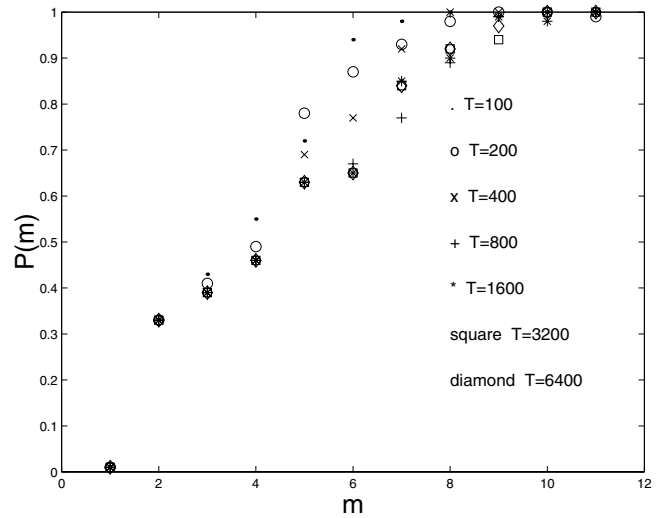


Fig. 3. Probability $P(m)$ for the market-maker to arbitrage the group of agents using (3)-(5) as a function of the memory length m . $P(m)$ is determined from the market-maker wealth after T time steps and by averaging over 100 simulations with different initial configurations. The parameters used are $N = 101, s = 5$. Similar results are found using different N, s .

a risk-return profile which should be unstable in the sense of standard economic theory. It is however a robust and stable feature of our model. This property results fundamentally from the heterogeneity of the strategies and can not be captured by a representative agent theory.

To study further the competition between the agents as a group and the market-maker, we let the $\$$ -game evolve for T time steps and measure if the market-maker has arbitrated the agents, *i.e.*, if his wealth is positive at the end of the time period T . Figure 3 shows the probability $P(m)$ for the market-maker to arbitrage the agents *versus* the memory of the agents m . For $m = 1$, the agents always exploit the market-maker according to the positive feedback mechanism involving the “stubborn majority” described above. As m increases, $P(m)$ increases and, for the largest memory $m = 11$ of the agents, the market-maker arbitrages the group of agents with probability 1. This correspond to the examples illustrated in Figure 2. In between, there is a competition between cooperativity between the agents and the destructive interferences of their heterogeneous strategies. The finite-size study of $P(m)$ as a function of T suggests the existence of a sharp transition in the large T limit for $m \approx 9$. Below this memory length, the set of strategies available to agents allow them to sometimes cooperate successfully. As the complexity of the information increases, their strategies are unable to cope with the large set of incoming information and the chaotic desynchronized behavior that results favors the market maker. This could be termed the curse of intelligence.

We will report elsewhere on extensions of this model with traders who act at different time scales and with different weights and on the detection of large price movements in the spirit of [18].

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